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EFFECTS OF FIBER-END CRACKS ON THE STIFFNESS OF ALIGNED SHORT-FIBER COMPOSITES†

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Abstract—The longitudinal Young's modulus of an aligned short-fiber reinforced composite with fiber-end cracks extending into the matrix is predicted theoretically in this paper. The analytical technique is based upon a modified Eshelby's equivalent inclusion method where infinite number of three kinds of ellipsoidal inhomogeneities are embedded in the matrix. The results indicate the importance of two parameters in affecting the stiffness of the composite: the size of the fiber-end crack, and the ratio of the number of fibers with fiber-end cracks to the total number of fibers.

1. INTRODUCTION

In short-fiber composites microcracks have been observed at various places: in the matrix, in the fibers, and at the fiber-ends. Microcracks play an important role in affecting the stiffness and strength of short-fiber composites[1]. The stress–strain relations of the short-fiber composites as affected by the existence of these microcracks have been discussed in [1, 2]. Taya and Mura[3] recently also studied the stiffness and strength of aligned short-fiber composites containing fiber-end cracks. In their model they assumed that the fiber-end crack is penny-shaped and its radius is as small as the radius of the fiber. The assumption that the fiber-end cracks are of small size was made because of the complexity of the geometry of the problem.

In this paper we are concerned with the case where the fiber-end cracks have extended into the matrix material, and hence the radius of the crack is larger than that of the fiber. We predict the overall longitudinal Young's modulus of the composite weakened by those extended fiber-end cracks. To this end we assume that fibers are aligned along the uniaxial loading directions and fiber-end cracks are penny-shaped. The fibers and fiber-end cracks are assumed to be ellipsoidal. The extension of the fiber-end cracks into the matrix has rendered the fiber segments near the ends ineffective in carrying axial load. Consequently, we reason that the contribution of a fiber with fiber-end cracks to the stiffness of the composition can be approximated by that of a shorter fiber with no cracks attached to its ends. As a result of this assumption, the present problem can be reduced to an analysis of three types of inhomogeneities in a matrix: perfect fibers, cracks and shortened fibers. Related to this treatment of inhomogeneity, we mention that the problem of two kinds of inhomogeneities has been formulated by Taya and Chou[4] within the framework of Eshelby's equivalent inclusion method[5]. By use of this formulation Taya[6] has also studied the case where penny-shaped cracks are located in the matrix. We will formulate the problem in Section 2 and present the results and discussions in Section 3. The conclusions are given in Section 4.

2. FORMULATION

We first describe the formulation for the general case of n kinds of ellipsoidal inhomogeneities in a matrix and apply it to the present problem. Consider an infinite elastic body containing infinite number of n kinds of ellipsoidal inhomogeneities and subjected to the applied stress σ_0 as shown in Fig. 1. Let the domains of the infinite body and the inhomogeneities of the mth kind be denoted by D and Ω_m, respectively. Hence the domain of the matrix becomes $D - \sum_{m=1}^{n} \Omega_m$. Denote the elastic stiffness tensors of the matrix and the inhomogeneity of the mth kind (Ω_m) by $\underset{\sim}{C}_0$ and $\underset{\sim}{C}_m$, respectively. The underneath tilda stands for tensorial quantities and the rank of the tensor should be self-explained.

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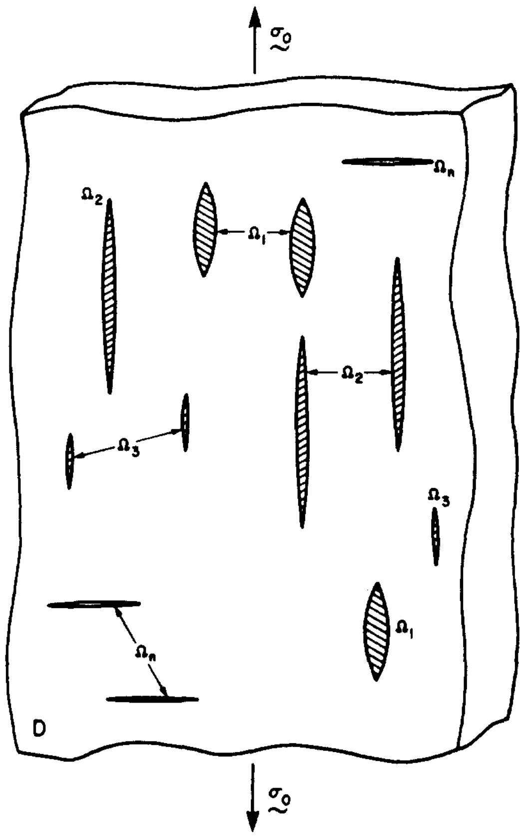

Fig. 1. Infinite number of n kinds of inhomogeneities embedded in an infinite body and subjected to the applied stress $\boldsymbol{\sigma}_0$.

Under the applied stress $\boldsymbol{\sigma}_0$ the average of the total stress in the matrix of the composite can be given by $\boldsymbol{\sigma}_0 + \langle\boldsymbol{\sigma}\rangle_m$, where

$$\langle\boldsymbol{\sigma}\rangle_m = \mathbf{C}_0\bar{\mathbf{e}}. \tag{1}$$

In the above equation $\langle\,\rangle$ denotes the volume averaged quantity and $\bar{\mathbf{e}}$ stands for the average disturbance in strain of the matrix due to all inhomogeneities $(\Omega_1, \Omega_2 \ldots \Omega_n)$.

According to Mori and Tanaka[7], we introduce a single inhomogeneity of the mth kind into the composite D. Then Eshelby's equivalent inclusion method yields[3–5]

$$\begin{aligned}\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_m &= \mathbf{C}_0(\mathbf{e}_0 + \bar{\mathbf{e}} + \mathbf{S}_m\mathbf{e}^{*m} - \mathbf{e}^{*m}) \\ &= \mathbf{C}_m(\mathbf{e}_0 + \bar{\mathbf{e}} + \mathbf{S}_m\mathbf{e}^{*m})\end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}_m$ is the disturbance of stress due to the presence of this single inclusion of the mth kind, and $\mathbf{S}_m$ is the Eshelby tensor of rank four for Ω_m. Also, $\mathbf{e}^{*m}$ is the eigenstrain (or transformation strain) which has non-vanishing components in Ω_m, but becomes zero outside of Ω_m. The stress disturbance $\boldsymbol{\sigma}_m$ in Ω_m can be obtained from eqn (2).

$$\boldsymbol{\sigma}_m = \mathbf{C}_0(\bar{\mathbf{e}} + \mathbf{S}_m\mathbf{e}^{*m} - \mathbf{e}^{*m}) \tag{3}$$

while it is understood

$$\boldsymbol{\sigma}_0 = \mathbf{C}_0\mathbf{e}_0. \tag{4}$$

Since the added single inhomogeneity of the mth kind can represent any single Ω_m, eqns (2) and (3) hold for any inclusion phase. The disturbance of the stress σ_m must satisfy $\int_D \sigma_m \, dV = 0$[4, 8]. Thus we obtain from eqns (1) and (3)

$$\left(1-\sum_{m=1}^{n} f_m\right)C_0\bar{e} + C_0 \sum_{m=1}^{n} f_m(\bar{e} + S_m e^{*m} - e^{*m}) = 0 \tag{5}$$

where f_m is the volume fraction of Ω_m. Premultiplying eqn (5) by C_0^{-1} we arrive at

$$\bar{e} + \sum_{m=1}^{n} f_m(S_m - I)e^{*m} = 0 \tag{6}$$

where I denotes the unit matrix. There are $n+1$ unknowns in this problem, $\bar{e}$ and e^{*m} ($m = 1 \sim n$), which will be solved by $n+1$ linear algebraic equations, eqns (2) and (6).

The equivalency of the strain energy of the composite yields[3–5]

$$\frac{1}{2}\sigma_0 C_c^{-1}\sigma_0 = \frac{1}{2}\sigma_0 C_0^{-1}\sigma_0 + \frac{1}{2}\sum_{m=1}^{n} f_m \sigma_0 e^{*m} \tag{7}$$

where C_c is the overall stiffness tensor of the composite to be computed. We consider here the uniaxially applied stress σ_0 along the x_3-axis as shown in Fig. 3. Then the overall longitudinal Young's modulus E_L of the composite can be obtained after having solved for e^{*m} and used eqn (7)

$$\frac{E_L}{E_0} = \frac{1}{1 + \sum_{m=1}^{n} f_m \beta_m} \tag{8}$$

where E_0 is the Young's modulus of the matrix and $\beta_m = e_{33}^{*m} E_0/\sigma_0$, where e_{33}^{*m} is the normal eigenstrain along the x_3-axis.

The above solution procedure is now applied to the present problem. The composite system with perfect fibers and fibers with fiber-end cracks (Fig. 2a) is converted to the problem of three

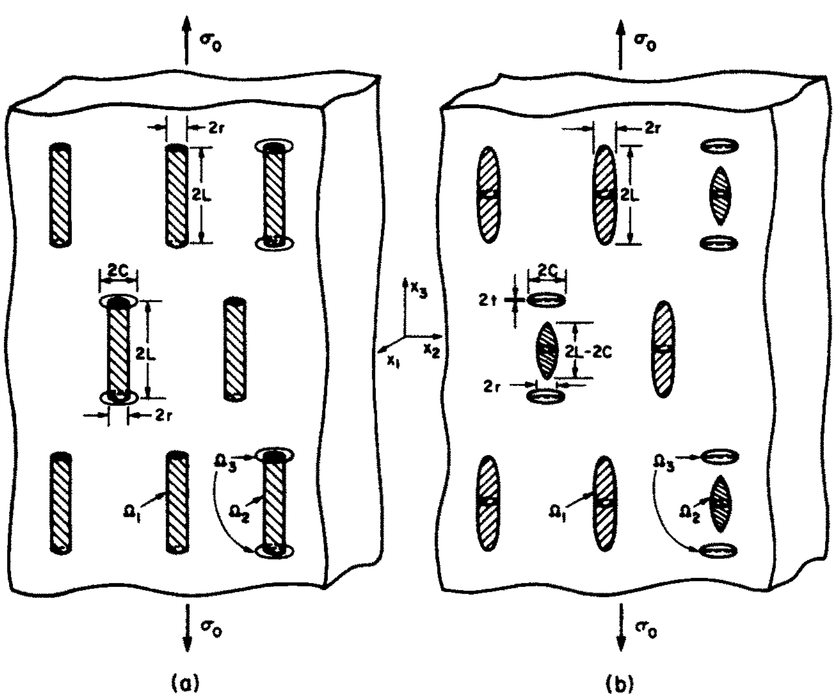

Fig. 2. (a) A model for an aligned short-fiber reinforced composite containing fiber-end cracks. (b) A calculation model for three kinds of inhomogeneities embedded in the matrix.

kinds of inhomogeneities (Fig. 2b). They include perfect fibers (Ω_1), fibers damaged by fiber-end cracks (Ω_2) and fiber-end cracks (Ω_3). It is assumed in Fig. 2(b) that the effective length of the damaged fiber is $2L$–$2c$ where $2L$ and $2c$ are the length of the perfect fiber and the diameter of penny-shaped crack, respectively. The above assumption is due to the fact that the distribution of σ_{33} (along the loading direction perpendicular to the penny-shaped crack) can be simulated by a constant stress for $|x_3| \leq L-c$. The exact solution[9] of a single crack embedded (the plane of the crack is along the x_1- and x_2-axes) in an infinite body subjected to applied stress along the x_3-axis, σ_0 indicates that $\sigma_{33} = 0.5\sigma_0$ at $|x_3| \cong c$, $\sigma_{33} = 0$ at $|x_3| = 0$ and its value approaches σ_0 as $|x_3|$ becomes large where $x_3 = 0$ corresponds to the origin of the crack. The fibers Ω_1 and Ω_2 are assumed to be prolate spheroids with the major and minor axes being L and r, and L–c and r, respectively, and are all aligned in the uniaxial loading direction (x_3-axis). The major and minor axes of the ellipsoidal penny-shaped crack are denoted by c and t, respectively and t is assumed to be infinitesimal, $t \ll c$. The system of Fig. 2 gives rise to transverse isotropy, hence each eigenstrain $\mathbf{e}^{*m}$ has two non-vanishing components, $e_{11}^{*m} = e_{22}^{*m}$ and e_{33}^{*m}. Thus we have 8 unknowns in the present problem. A computer program has been used to solve for the unknowns numerically. The eigenstrain e_{33}^{*3} in the penny-shaped crack can have the following form[3]:

$$e_{33}^{*3} = \alpha \cdot \frac{c}{t}\frac{\sigma_0}{E_0} \tag{9}$$

where the coefficient α will be computed numerically. When only one penny-shaped crack is embedded in an infinite body subjected to the applied stress σ_0 in the x_3-direction and is perpendicular to the loading direction, α is equal to $4(1-\nu_0^2)/\pi$[3, 8] where ν_0 is the Poisson's ratio of the matrix. The equation to compute the overall longitudinal Young's modulus E_L is now reduced to

$$\frac{E_L}{E_0} = \frac{1}{\left\{1 + f_1\beta_1 + f_2\beta_2 + \frac{2\alpha c3}{(L-c)r^2} f_2\right\}} \tag{10}$$

where f_1 and f_2 are the volume fractions of the perfect fibers and damaged fibers whose length is $2L-2c$ respectively, and $f = f_1 + f_2[L/(L-c)]$ is the volume fraction of fibers before any fiber-end crack appears. Where $f_D f = f_2[L/(L-c)]$ is the volume fraction of the damaged fibers before they are chopped to their length $2L-2c$ and f_D is the ratio of the number of the damaged fibers to that of all fibers. In other words $f_D = 0$ indicates that all fibers are perfect and for $f_D = 1$ all fibers yield fiber-end cracks.

3. RESULTS AND DISCUSSIONS

The mechanical properties of short carbon fiber reinforced polyamide 66 are used for our computation: $E_0 = 2\times 10^9$ N/m^2, $E_f = 2\times 10^{11}$ N/m^2, $\nu_0 = 0.42$, and $\nu_f = 0.17$, where the subscript f denotes the fiber. Also, the fiber aspect ratio $L/r = 50$ and the total fiber volume fraction $f = 0.2$[1].

The longitudinal Young's modulus E_L of the composite containing fiber-end cracks is computed from eqn (10) and is plotted as a function of the ratio of the crack radius to the fiber radius, c/r in Fig. 3. Here, f_D is taken as a parameter ranging from 0 to 1, and E_L is normalized by E_*, the longitudinal Young's modulus of the composite without fiber-end cracks. It follows from Fig. 3 that E_L reduces rapidly as the crack size increases in the range of $c/r = 3 \sim 10$, thereafter the rate of the reduction in E_L becomes smaller. E_L also reduces as f_D increases. To see this more clearly E_L/E_* is plotted as a function of f_D in Fig. 4 where c/r is taken as a parameter. It is clear from this figure that the reduction in E_L becomes more significant for relatively large values of c/r and small values of f_D.

We have also computed E_L for the case where the damaged fibers are neglected in the analysis, i.e. Ω_2 is replaced by the matrix material, and the results are plotted by dashed curves in Fig. 5 for $f_D = 0.1$ and 1.0. The analysis neglecting the damaged fibers gives conservative estimate of E_L and can be used as a good approximation of E_L for large c/r and small f_D values.

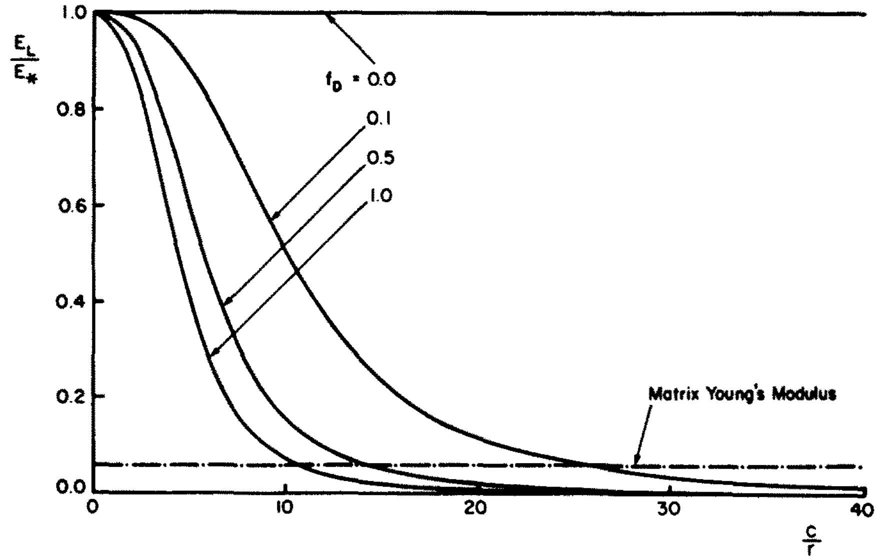

Fig. 3. The longitudinal Young's modulus E_L vs the crack size c for the ratio of the damaged fibers to the total number of fibers, $f_D = 0, 0.1, 0.5$ and 1.0. E_* and r are E_L of the composite without cracks and the radius of the fiber, respectively.

However, when c/r is less than 10 and $f_D \to 1$, the above simple analysis is no longer valid. The result given by the black circle in Fig. 5 is obtained by Taya and Mura[4] for the case of $c/r = 1.0$. It was assumed in their model that the penny-shaped cracks are in contact with the ellipsoidal fiber at its end and the unknown eigenstrain e_{ij}^{*c} in the crack was distributed uniformly in the domain. The eigenstrain e_{ij}^{*c} was computed such that the total stress in the crack vanished, $\sigma_{ij}^0 + \bar{\sigma}_{ij}^f(e_{ij}^{*f}) + \sigma_{ij}^c(e_{ij}^{*c}) = 0$. Here, $\bar{\sigma}_{ij}^f$ is the disturbed stress just outside of the fiber, σ_{ij}^c is the disturbed stress in the crack and e_{ij}^{*f} is the eigenstrain in the fiber. The eigenstrain e_{33}^{*c} so computed is accurate at the contact point, but may be overestimated in the crack domain other than the contact point. The overestimated value of e_{33}^{*n} reduces the value of E_L[4]. On the other hand, the effective length of the fiber (Ω_2) in the present model is $2(L-c)$ and has almost the same effect on E_L as the perfect fiber does when $L \gg c$ (the present case, $L/c = 50$). Thus the present model tends to overestimate E_L. The above two reasons are believed to be responsible for the fact that E_L computed by the present model exceeds that by the previous model[4] for $c/r = 1.0$. In the Figs. 3–5, the chain-dot lines stand for the matrix Young's modulus.

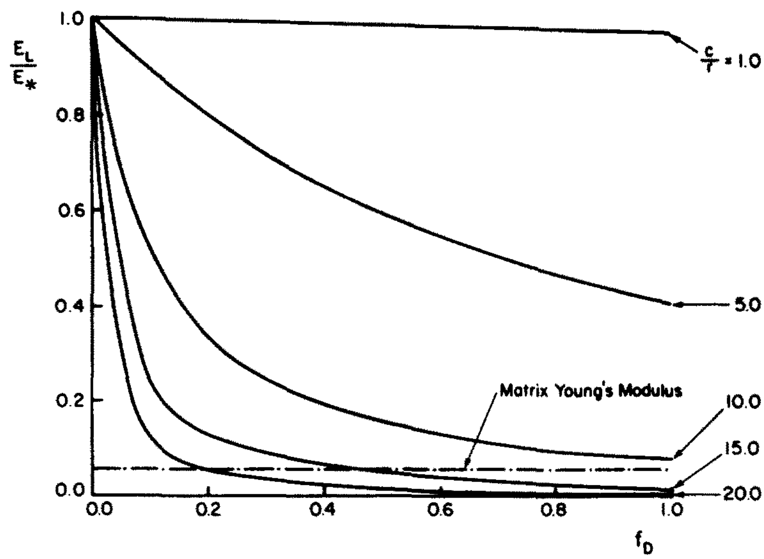

Fig. 4. E_L/E_* vs f_D for $c/r = 1, 5, 10, 15$ and 20.

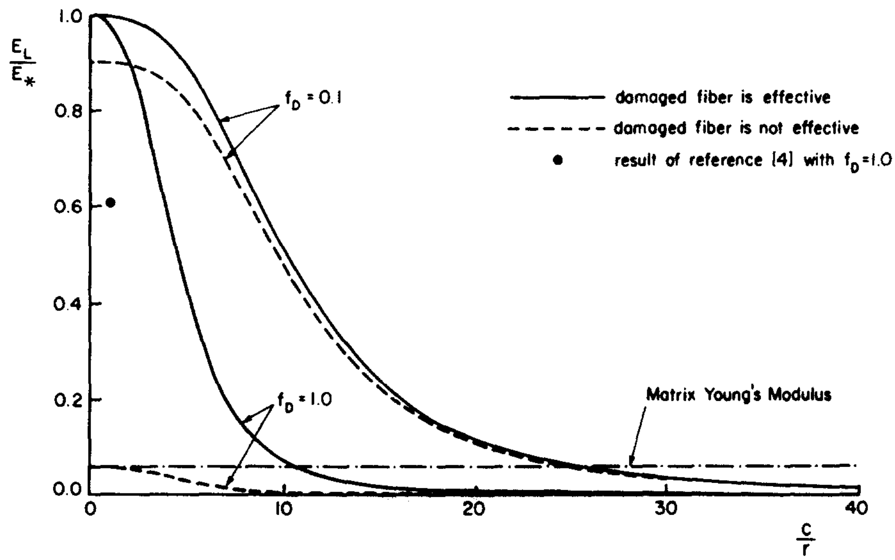

Fig. 5. E_L of the composite with the damaged fiber being effective (——) and not effective (----) for $f_D = 0.1$ and 1.0 vs c/r. The black circle result was obtained from Ref. [4] for $c/r = 1.0$ and $f_D = 1.0$.

4. CONCLUSIONS

The longitudinal Young's modulus of an aligned short-fiber composite is computed when it contains fiber-end cracks which have propagated into the matrix. The present work extends that of Taya and Mura[4], who focused on the case of small fiber-end cracks. The following conclusions can be made:

(1) As the fiber-end cracks extending into the matrix, the reduction on composite Young's modulus is relatively more significant during the initial period, say $c/r = 1 \sim 10$, than when $c/r > 10$.

(2) The ratio of the number of fibers that have developed fiber-end cracks to the total number of fibers in the composite, f_D, has relatively more significant influence on composite Young's modulus when c/r is large and f_D itself is small.

(3) The analysis neglecting the contribution of fibers with fiber-end cracks is a good approximation to the composite Young's modulus for $c/r \geqslant 10$ at small f_D values and for $c/r \geqslant 20$ at large f_D values.

REFERENCES

1. R. T. Curtis, M. G. Bader and J. E. Bailey, The stiffness and strength of a polyamide thermoplastic reinforced with glass and carbon fibers. *J. Material Sci.* **13**, 377–390 (1978).
2. M. Taya and T. W. Chou, Prediction of the first and second stages of stress–strain curves of unidirectional short fiber reinforced thermoplastics. *Proc. Japan–U.S. Conf. on Composite Materials: Mechanics, Mechanical Properties and Fabrication*, 12–14 *Jan.*, 1981, Tokyo, pp. 119–130.
3. M. Taya and T. Mura, On stiffness and strength of an aligned short-fiber reinforced composite under uniaxial applied stress when the composite contains fiber-end cracks. *J. Appl. Mech.* **48**, 361–367 (1981).
4. M. Taya and T. W. Chou, On two kinds of ellipsoidal inhomogeneities in an infinite elastic body: an application to a hybrid composite. *Int. J. Solids Structures* **17**, 553–563 (1981).
5. J. D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. R. Soc. (London)* **A241**, 376–396 (1957).
6. M. Taya, On stiffness and strength of an aligned short-fiber reinforced composite containing penny-shaped cracks in the matrix. *J. Composite Materials* **15**, 198–210 (1981).
7. T. Mori and K. Tanaka, Average stress in matrix and average elastic energy of materials with misfitting inclusions. *Acta Metallurgica* **21**, 571–574 (1973).
8. T. Mura, *Micromechanics of Defects in Solids*. In press. Noordhoff, Leyden.
9. H. M. Westergaard, Bearing pressures and cracks. *J. Appl. Mech.* **6**, 49–53 (1939).